The path integral for Chern-Simons quantum mechanics

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Abstract

The path integral representation for a system of N non-relativistic particles on the plane, interacting through a Chern-Simons gauge field, is obtained from the operator formalism. An effective interaction between the particles appears, generating the usual Aharonov-Bohm phases. The spin-statistics relation is also considered.

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The unifying concept for planar systems with fractional statistics is the long range, non-local mutual interaction of particles carrying an U(1) charge e with "statistical" flux tubes at their location, in an Aharonov-Bohm effect scenario [1]. A local field theoretical description of these systems is provided if we couple the matter fields to a gauge field $A_\mu = (\mathbf{A}(\mathbf{x},t), A_0(\mathbf{x},\mathbf{t}))$ with the dynamics given by the topological Chern-Simons density [2]

$$\mathcal{L}(A) = \frac{\theta}{2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda\,. \tag{1}$$

This interaction has the effect of endowing the coupled particles with flux tubes of strength $-\frac{e}{\theta}$, and the statistics enters through the Aharonov-Bohm phase that appears when one particle encircles another, in addition to the intrinsic statistical one.

In this paper we obtain the path integral representation for the transition amplitude of a planar system with N non-relativistic particles, in interaction with a Chern-Simons gauge field. The effective action of this path integral displays both the self-induced spin and Aharonov-Bohm effects. To obtain this representation we perform the canonical quantization of this system and after that apply a method previously discussed in [3] for non-relativistic particles with background gauge fields.

The Lagrangian for this planar system with N non-relativistic particles, interacting with the Chern-Simons gauge field is given by

$$L = \sum_{a=1}^{N}\Big(\frac{m}{2}\dot{\mathbf{x}}_a^2 + e\dot{\mathbf{x}}_a\cdot\mathbf{A}(\mathbf{x}_a,t) - eA_0(\mathbf{x}_a,t)\Big) + \theta\int d^2x\Big(\frac{1}{2}\epsilon^{ij}\dot{A}_iA_j - A_0B\Big)\,. \tag{2}$$

In order to quantize the above system we proceed by defining the momenta for the point particles as $p_a^i \equiv \partial L/\partial\dot{x}_a^i$ and passing to the Hamiltonian formalism. The corresponding quantum operators and canonical commutation relations are given by ($\hbar = 1$)

$$[\hat{A}_i(\mathbf{x},t), \hat{A}_j(\mathbf{y},t)] = \frac{i}{\theta}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y})\,, \tag{3}$$

$$[\hat{x}_i^a, \hat{x}_j^b] = 0\,, \tag{4}$$

$$[\hat{x}_i^a, \hat{\pi}_j^b] = i\delta^{ab}\delta_{ij}\,, \tag{5}$$

$$[\hat{\pi}_i^a, \hat{\pi}_j^b] = ie\epsilon_{ij}[\hat{B}(\hat{\mathbf{x}}_b)\delta_{ab} + \frac{e}{\theta}\delta(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b)]\,, \tag{6}$$

with $\hat{\pi}^i_a \equiv \hat{p}^i_a - e\hat{A}^i(x_a)$ and $\hat{B} = \epsilon_{ij}\partial_i\hat{A}_j$. The $\hat{A}_0$ field has zero conjugate momentum being so a Lagrange multiplier. The time evolution is generated by the Hamiltonian operator

$$\hat{H} = \sum_{a=1}^{N}\left(\frac{\hat{\boldsymbol{\pi}}_a{}^2}{2m} + e\hat{A}_0(x_a,t)\right) + \theta\int d^2x\hat{A}_0(x,t)\hat{B}(x,t)\,. \tag{7}$$

Introducing the eigenvectors of $\hat{\mathbf{x}}_a(t)$ defined by $\hat{\mathbf{x}}_a(t)|x',t\rangle = x'_a|x',t\rangle$, and the Schrödinger wave functional $\Psi[A,t)$ for the gauge fields we want to consider the transition amplitude

$$G(\mathbf{x}'',\mathbf{x}';T) = \int[d\mu(A)]\Psi[A,0)^*\langle\mathbf{x}''|e^{-i\hat{H}T}|\mathbf{x}'\rangle\Psi[A,0) \tag{8}$$

where $|x',0\rangle \equiv |x'\rangle$ and $[d\mu(A)]$ is the integration measure over all the configurations of the gauge field A_μ. To obtain the wave functional $\Psi[A,t)$ we must decide on which components of the gauge field it depends since they do not commute with one another. We start with a convenient decomposition of the Schrödinger representation field [1] $\hat{\mathbf{A}}(\mathbf{x})$ into longitudinal and transverse parts

$$\hat{A}_i(\mathbf{x}) = \partial_i\hat{\xi}(\mathbf{x}) - \epsilon_{ij}\frac{\partial_j}{\nabla^2}\hat{B}(\mathbf{x})\,. \tag{9}$$

With this decomposition we have that (3) leads to

$$[\hat{\xi}(\mathbf{x}),\hat{B}(\mathbf{y})] = -\frac{i}{\theta}\delta(\mathbf{x}-\mathbf{y})\,, \tag{10}$$

Now we choose our wave functional to depend only on ξ since the conjugate momentum to A_0 is zero. $\hat{B}$ acts on it as the functional derivative $i\delta/\delta(\theta\xi)$. The functional Schrödinger equation for the gauge field coupled to N point-like sources as one can read from (7) is [4]

$$i\partial_t\Psi[\xi,t) = \int d^2x\left[i\left(A_0 + \frac{1}{\theta}J_i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\right)\frac{\delta}{\delta\xi} + A_0\rho - \xi\dot{\rho}\right]\Psi[\xi,t)\,, \tag{11}$$

with $J_i(x,t) = e\sum_{a=1}^{N}\dot{x}^a_i(t)\delta(\mathbf{x}-\mathbf{x}_a(t))$ and $\rho(x,t) = e\sum_{a=1}^{N}\delta(\mathbf{x}-\mathbf{x}_a(t))$. The solution is given by

[1] hereafter the fields with no time argument are evaluated at t=0

$$\Psi[\xi,t)=exp\Big[i\int d^2x\Big(\xi(\mathbf{x})\rho(\mathbf{x},t)+\int^t d\tau J_i(\mathbf{x},\tau)\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho(\mathbf{x},\tau)\Big)\Big] \tag{12}$$

As we can see the above wave functional satisfies the Gauss law

$$\Big[\hat{B}(\mathbf{x})+\frac{e}{\theta}\sum_{a=1}^{N}\delta(\mathbf{x}-\mathbf{x}_a(t))\Big]\Psi[\xi,t)=0\,, \tag{13}$$

fixing the possible eigenvalues of the conjugate momentum of $\hat{\xi}$. Since both $\hat{A}_0$ and $\hat{\xi}$ momenta are constrained we must, in order to obtain a scalar product for $\Psi[\xi,t)$, choose an integration measure $[d\mu(\xi,A_0)]$ that selects one particular configuration of the fields. Our choice is $[d\xi][dA_0]\delta[\xi-\xi']\delta[A_0-A_0']$, where ξ' and A_0' are arbitrary functions of x, that for the sake of simplicity we set equal to zero everywhere. With this wave functional at hand we proceed to write a path integral for the operator valued kernel $G(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})\equiv\langle\mathbf{x}''|e^{-i\hat{H}T}|\mathbf{x}'\rangle$, decomposing it as [5]

$$G(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})=\int d\mathbf{x}_p\ldots\int d\mathbf{x}_1\langle\mathbf{x}'',T|\mathbf{x}_p,t_p\rangle\langle\mathbf{x}_p,t_p|\mathbf{x}_{p-1},t_{p-1}\rangle\ldots\langle\mathbf{x}_1,t_1|\mathbf{x}',0\rangle\,. \tag{14}$$

Then we take $T=(p+1)\varepsilon$ and $t_n=n\varepsilon$ $(n=1,\ldots,p)$ letting $p\to\infty$ and $\varepsilon\to 0$ with T fixed.

To evaluate the small time propagator $\langle\mathbf{x}_{n+1},t_n+\varepsilon|\mathbf{x}_n,t_n\rangle$ we use a method developed by Schwinger in his early investigations on effective actions [6]. We write the kernel $G(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})$ as $exp(iW)$, where $W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})$ is a complex functional of $\hat{\mathbf{A}}$, the end point coordinates and time. Defining the expectation value of an observable $\hat{\mathcal{O}}$ by

$$\langle\hat{\mathcal{O}}\rangle\equiv\langle\mathbf{x}'',T|\hat{\mathcal{O}}|\mathbf{x}',0\rangle/\langle\mathbf{x}'',T|\mathbf{x}',0\rangle, \tag{15}$$

it is easy to verify that W is determined by the following equations

$$\langle\hat{H}(\hat{\mathbf{x}}(T),\hat{\boldsymbol{\pi}}(T))\rangle=-\frac{\partial W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})}{\partial T}, \tag{16}$$

$$\langle\hat{\pi}_i^a(T)\rangle=\frac{\partial W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})}{\partial x''_{i,a}}+\frac{e}{\theta}\epsilon_{ij}\frac{\delta W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})}{\delta\hat{A}_j(\mathbf{x}''_a)}-e\hat{A}_i(\mathbf{x}''_a), \tag{17}$$

$$\langle\hat{\pi}_i^a(0)\rangle=-\frac{\partial W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})}{\partial x'_{i,a}}-\frac{e}{\theta}\epsilon_{ij}\frac{\delta W(\mathbf{x}'',\mathbf{x}';T|\hat{\mathbf{A}})}{\delta\hat{A}_j(\mathbf{x}'_a)}-e\hat{A}_i(\mathbf{x}'_a), \tag{18}$$

$$W(\mathbf{x}'',\mathbf{x}';0|\hat{\mathbf{A}})=-i\,ln\delta^{2N}(\mathbf{x}''-\mathbf{x}'). \tag{19}$$

To solve this problem Schwinger noticed that the above equations relate the transition amplitude to the solution of the Heisenberg equations for $\hat{\mathbf{x}}_a(T)$ and $\hat{\boldsymbol{\pi}}_a(T)$. If we solve for $\hat{\boldsymbol{\pi}}_a(T)$ in terms of $\hat{\mathbf{x}}_a(T)$ and $\hat{\mathbf{x}}_a(0)$ and insert this, in a time ordered fashion, in (16)-(18) we are left with a set of first order equations to integrate. For the small time kernel we must have $\hat{\mathbf{x}}_a(\varepsilon)$ up to second order in the small time ε

$$\hat{x}^i_a(\varepsilon) = e^{i\hat{H}\varepsilon}\hat{x}^i_a(0)e^{-i\hat{H}\varepsilon} \simeq \hat{x}^i_a(0) + \frac{\hat{\pi}^i_a(0)}{m}\varepsilon + \frac{e}{2m^2}\sum_{b=1}^{N}\epsilon^{ij}[\hat{B}(\hat{\mathbf{x}}_b)\delta_{ab} + \frac{e}{\theta}\delta(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b)]\hat{\pi}^j_b(0)\varepsilon^2 \quad (20)$$

Inverting the above equation to get $\hat{\pi}^i_a(0)$ in terms of $\hat{\mathbf{x}}_a(\varepsilon)$ and $\hat{\mathbf{x}}_a(0)$

$$\hat{\pi}^i_a(0) \simeq m\frac{\Delta\hat{x}^i_a}{\varepsilon} - \frac{e}{2}\sum_{b=1}^{N}\epsilon^{ij}[\hat{B}(\hat{\mathbf{x}}_b(\varepsilon))\delta_{ab} + \frac{e}{\theta}\delta(\hat{\mathbf{x}}_a(\varepsilon) - \hat{\mathbf{x}}_b(\varepsilon))]\Delta\hat{x}^j_b\,, \quad (21)$$

where $\Delta\hat{x}^i_a = \hat{x}^i_a(\varepsilon) - \hat{x}^i_a(0)$. Using the fact that $\langle(\Delta\hat{\mathbf{x}}_a)^2\rangle$ is of order ε [5], we see that if we take $\langle\hat{\boldsymbol{\pi}}_a(0)\rangle$ the terms in the above expansion are respectively of order $1/\sqrt{\varepsilon}$ and $\sqrt{\varepsilon}$. The second term although small in comparison with the first, will give a relevant contribution when used in (17) and (18) to evaluate $W(\mathbf{x}_{n+1}, \mathbf{x}_n; \varepsilon|\hat{\mathbf{A}})$. From $\hat{\boldsymbol{\pi}}_a(0)$ we have by time evolution

$$\hat{\pi}^i_a(\varepsilon) \simeq m\frac{\Delta\hat{x}^i_a}{\varepsilon} + \frac{e}{2}\sum_{b=1}^{N}\epsilon^{ij}[\hat{B}(\hat{\mathbf{x}}_b(\varepsilon))\delta_{ab} + \frac{e}{\theta}\delta(\hat{\mathbf{x}}_a(\varepsilon) - \hat{\mathbf{x}}_b(\varepsilon))]\Delta\hat{x}^j_b\,. \quad (22)$$

Using the above expression for $\hat{\boldsymbol{\pi}}_a(0)$ or $\hat{\boldsymbol{\pi}}_a(\varepsilon)$ in $\hat{H}$, in a time ordered manner, we are ready to integrate (16)

$$W(\mathbf{x}_{n+1}, \mathbf{x}_n; \varepsilon|\hat{\mathbf{A}}) \simeq \sum_{a=1}^{N}\frac{m(\mathbf{x}^{n+1}_a - \mathbf{x}^n_a)^2}{2\varepsilon} + iN\,ln\,\varepsilon + \Phi(\mathbf{x}_{n+1}, \mathbf{x}_n|\hat{\mathbf{A}})\,, \quad (23)$$

where we used that $[\hat{x}^i_a(\varepsilon), \hat{x}^j_b(0)] \simeq -i\delta^{ij}\delta_{ab}\varepsilon/m$ and Φ is a time independent functional of $\mathbf{A}$ and the end point coordinates $\mathbf{x}^{n+1}_a$ and $\mathbf{x}^n_a$. Inserting the above W in (17) and (18) we have for Φ (remember $\langle(\Delta\hat{\mathbf{x}}_a)^2\rangle \sim \varepsilon$)

$$\frac{\partial\Phi}{\partial x^{n+1}_{i,a}} + \frac{e}{\theta}\epsilon_{ij}\frac{\delta\Phi}{\delta\hat{A}_j(\mathbf{x}^{n+1}_a)} = \frac{e}{2}\Big[\Delta x^{k,n}_a\frac{\partial\hat{A}_k(\mathbf{x}^{n+1}_a)}{\partial x^{n+1}_{i,a}} + \hat{A}^i(\mathbf{x}^{n+1}_a) + \hat{A}^i(\mathbf{x}^n_a) \quad (24)$$

$$+ \frac{e}{\theta}\sum_{b=1}^{N}\epsilon^{ij}\delta^2(\mathbf{x}^{n+1}_a - \mathbf{x}^{n+1}_b)\Delta x^{j,n}_b\Big] + O(\varepsilon)\,, \quad (25)$$

$$\frac{\partial\Phi}{\partial x^n_{i,a}} + \frac{e}{\theta}\epsilon_{ij}\frac{\delta\Phi}{\delta\hat{A}_j(\mathbf{x}^n_a)} = \frac{e}{2}\Big[\Delta x^{k,n}_a\frac{\partial\hat{A}_k(\mathbf{x}^n_a)}{\partial x^n_{i,a}} - \hat{A}^i(\mathbf{x}^{n+1}_a) - \hat{A}^i(\mathbf{x}^n_a) \quad (26)$$

$$+ \frac{e}{\theta}\sum_{b=1}^{N}\epsilon^{ij}\delta^2(\mathbf{x}^n_a - \mathbf{x}^n_b)\Delta x^{j,n}_b\Big] + O(\varepsilon)\,, \quad (27)$$

with $\Delta \mathbf{x}_a^n = \mathbf{x}_a^{n+1} - \mathbf{x}_a^n$. Since in the rhs of the above equations we have $O(\sqrt{\varepsilon})$ terms we are allowed to discard the $O(\varepsilon)$ ones and easily find the solution for Φ

$$\Phi(\mathbf{x}_{n+1}, \mathbf{x}_n|\hat{\mathbf{A}}) = e \sum_{a=1}^{N} \Delta \mathbf{x}_n^a \cdot \hat{\mathbf{A}}(\bar{\mathbf{x}}^a) + C\,, \tag{28}$$

where $\bar{\mathbf{x}}^a = \frac{1}{2}(\mathbf{x}_a^{n+1} + \mathbf{x}_a^n)$ and C is the constant $-iN\,ln(m/2\pi i)$ determined by (19), setting $\varepsilon \to -i0_+$ in e^{iW} and using the Gaussian representation for the delta function.

Putting this all together we get the small time kernel

$$G(\mathbf{x}_{n+1}, \mathbf{x}_n; \varepsilon|\hat{\mathbf{A}}) \simeq \Big(\frac{m}{2\pi i\varepsilon}\Big)^N exp\Big[i \sum_{a=1}^{N}\Big(\frac{m(\Delta \mathbf{x}_n^a)^2}{2\varepsilon} + e\Delta \mathbf{x}_n^a \cdot \hat{\mathbf{A}}(\bar{\mathbf{x}}^a)\Big)\Big]\,. \tag{29}$$

Inserting it in (14) we get the functional integral representation for $G(\mathbf{x}'', \mathbf{x}'; T|\hat{\mathbf{A}})$,

$$G(\mathbf{x}'', \mathbf{x}'; T|\hat{\mathbf{A}}) = \int [d^N\mathbf{x}] exp\Big(i \sum_{a=1}^{N} \frac{m}{2} \dot{\mathbf{x}}_a^2\Big)\, \mathbf{T}exp\Big(ie \sum_{a=1}^{N} \int_0^T dt\, \dot{\mathbf{x}}_a \cdot \hat{\mathbf{A}}(\mathbf{x}_a)\Big)\,, \tag{30}$$

we here use the (abusive) continuum language where by $[d^N\mathbf{x}]$ we mean the infinite product of terms $(m/2\pi i\varepsilon)^N d^N\mathbf{x}_k$ and the integration runs from $\mathbf{x}(0) = \mathbf{x}'$ to $\mathbf{x}(T) = \mathbf{x}''$. $\mathbf{T}exp \int \hat{\mathbf{A}}$ denotes the time ordered exponential that stems naturally from (14). To undo this time ordering in order to insert (30) in (8) we use the BCH formula

$$exp(M_k)\, exp(M_{k-1}) \dots exp(M_0) = exp(\sum_n M_n)\, exp(\frac{1}{2} \sum_{m>n} [M_m, M_n])\,, \tag{31}$$

($[M_m, M_n]$=c-number) that gives,

$$\mathbf{T}exp\Big(ie \sum_{a=1}^{N} \int_0^T dt\, \dot{\mathbf{x}}_a \cdot \hat{\mathbf{A}}(\mathbf{x}_a)\Big) = exp(i\omega)\, exp\Big(ie \sum_{a=1}^{N} \int_0^T dt\, \dot{\mathbf{x}}_a \cdot \hat{\mathbf{A}}(\mathbf{x}_a)\Big)\,, \tag{32}$$

with

$$\omega = -\frac{e^2}{2\theta} \sum_{a,b=1}^{N} \int_0^T dt \int_0^t d\tau\, \dot{x}_a^i(t) \epsilon_{ij} \dot{x}_b^j(\tau)\, \delta(\mathbf{x}_a(t) - \mathbf{x}_b(\tau))\,. \tag{33}$$

Using the polarization (9) we can now evaluate the expectation value in (8),

$$\begin{aligned} &\int [d\mu(A)] \Psi[\xi, 0]^* \mathbf{T}exp\Big(ie \sum_{a=1}^{N} \int_0^T dt\, \dot{\mathbf{x}}_a \cdot \hat{\mathbf{A}}(\mathbf{x}_a)\Big) \Psi[\xi, 0] \\ &= \quad exp\, i\Big(\omega + \frac{e^2}{\theta} \sum_{a,b=1}^{N} \int_0^T dt\, \dot{x}_a^i \epsilon_{ij} \frac{\partial_a^j}{\nabla_a^2} \delta(\mathbf{x}_a(t) - \mathbf{x}_b(0))\Big)\,. \end{aligned} \tag{34}$$

The argument of the exponential in the last line of (34) is the effective interaction between the N particles. Noticing that

$$\omega = \frac{e^2}{\theta}\sum_{a,b=1}^{N}\int_0^T dt\, \dot{x}_a^i \epsilon_{ij}\frac{\partial_a^j}{\nabla_a^2}\Big(\delta(\mathbf{x}_a(t)-\mathbf{x}_b(t)) - \delta(\mathbf{x}_a(t)-\mathbf{x}_b(0))\Big)\,, \tag{35}$$

we have for the $a \neq b$ case the following interaction

$$\begin{aligned} S_{int} &= \frac{e^2}{\theta}\sum_{\substack{a,b=1\\ a\neq b}}^{N}\int_0^T dt\, \dot{x}_a^i \epsilon_{ij}\frac{\partial_a^j}{\nabla_a^2}\delta(\mathbf{x}_a(t)-\mathbf{x}_b(t)) \\ &= \frac{e^2}{2\pi\theta}\sum_{\substack{a,b=1\\ a\neq b}}^{N}\int_0^T dt\, \dot{x}_a^i \epsilon_{ij}\frac{(x_a-x_b)^j}{|\mathbf{x}_a-\mathbf{x}_b|^2}\,. \end{aligned} \tag{36}$$

That is just the ordinary long range Aharonov-Bohm interaction between the N particles. For $a = b$ we have the self-interaction terms

$$\begin{aligned} S_{self} &= \lim_{t\to t'}\frac{e^2}{\theta}\sum_{a=1}^{N}\int_0^T dt\, \dot{x}_a^i \epsilon_{ij}\frac{\partial_a^j}{\nabla_a^2}\delta(\mathbf{x}_a(t)-\mathbf{x}_a(t')) \\ &= -\frac{e^2}{4\pi\theta}\sum_{a=1}^{N}\int_0^T dt\, \frac{d}{dt} tan^{-1}\Big(\frac{\dot{x}_2^a(t)}{\dot{x}_1^a(t)}\Big)\,. \end{aligned} \tag{37}$$

To arrive at the last line we used the fact that $\epsilon_{ij}\partial_i\partial_j tan^{-1}(\mathbf{x}) = 2\pi\delta(\mathbf{x})$ and the identity [4]

$$\lim_{t\to t'}\frac{d}{dt'}tan^{-1}\Big(\frac{x_2(t')-x_2(t)}{x_1(t')-x_1(t)}\Big) = \frac{1}{2}\frac{d}{dt}tan^{-1}\Big(\frac{\dot{x}_2(t)}{\dot{x}_1(t)}\Big)\,. \tag{38}$$

The quantity in the last line of (37) is proportional to the sum of angles swept by $\dot{\mathbf{x}}_a(t)$ along the paths. Take for example N=1, for a closed smooth path with no self-intersections we have a phase factor $exp\, iS_{self} = exp\, i(-\frac{e^2}{2\theta}) = exp\, i\, 2\pi s$, with s defined as the spin of the particle, giving the celebrated spin-statistics relation $|s| = \frac{e^2}{4\pi\theta}$. We achieved this result with no regulator for the δ function, if we had used a regulator that preserves the odd nature of $\partial_i\delta(\mathbf{x})$ (e.g. Gaussian) such that $\partial_i\delta(0) = 0$ [4], there would be no spin contribution at all. This ultraviolet ambiguity is a shortfall of the calculations involving the Chern-Simons term. In a recent paper [7] this question was investigated, by avoiding the Chern-Simons term, in a Berry phase calculation and no spin-statistics relation was found on the plane, for the fractional quantum Hall effect quasiparticles.

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